A Note on the relationship between 1^{st}-Order and 2^{nd}-Order Wave Equations

Consider the one-way wave equation:

$$(\partial_x \pm c_0^{-1}\partial_t)u = 0$$

$$u = f\left(t \mp \frac{x}{c_0}\right)$$

But:

$$\left(\partial_x + c_0^{-1}\partial_t\right)\left(\partial_x - c_0^{-1}\partial_t\right)$$

$$\partial_x^2 + c_0^{-1}\partial_{tx} - c_0^{-1}\partial_{xt} - c_0^{-2}\partial_t^2 = 0$$

$$= \partial_x^2 - c_0^{-2}\partial_t^2 \quad (2^{nd}\text{ Order Wave Eq.})$$

$$\therefore \quad \left(\partial_x^2 - c_0^{-2}\partial_t^2\right) f\left(t \pm \frac{x}{c_0}\right) = 0$$

that's why the general solution to the 2^{nd}-Order wave equation is:

$$u(x,t) = f(t + x/c_0) + g(t - x/c_0)$$

so $(\partial_x + c_0^{-1}\partial_t)u(x,t) = 0 \Rightarrow f(t - x/c_0)$

$(\partial_x - c_0^{-1}\partial_t)u(x,t) = 0 \Rightarrow f(t + x/c_0)$

so at the $x=0$ boundary, a first-order wave equation is

$$(\partial_x - c_0^{-1}\partial_t)u = 0$$

but we don't know the velocity in the x-direction.

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Absorbing Boundary Conditions

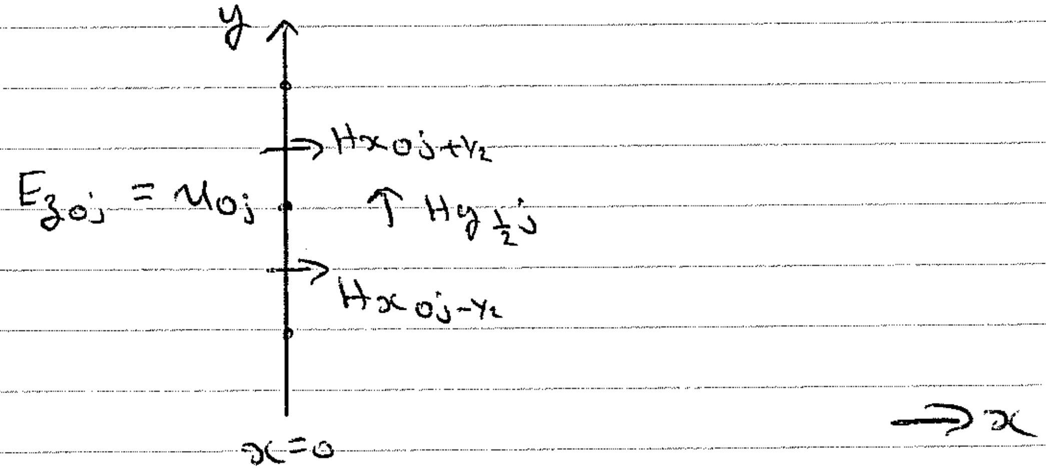
Consider a single component of $\bar{E}$ or $\bar{H}$, say u. It will satisfy the 2nd order wave equation:

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2 - c_0^{-2}\partial_t^2\right) u = 0$$

where c_0 is the speed of propagation.

Now on our FDTD grid we won't be able to apply our normal centred difference method at the boundaries for some components.

For example, conside the TM case in 2D:

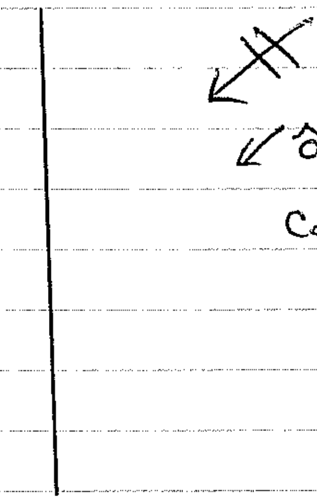


We will be able to update $Hx_{0j+1/2}$ and $Hx_{0j-1/2}$ but not Ez_{0j} because it depends on

$$Hx_{0j+1/2} \quad Hx_{0j-1/2} \quad Hy_{\frac{1}{2}j} \quad \underline{\text{AND}} \quad \boxed{Hy_{-\frac{1}{2}j}}$$

One way to come up with ABCs is to assume that the field is a plane wave impinging on the boundary at a certain angle.

Sp we're interested in the boundary $x=0$



plane-wave

$\hat{a}$ - direction of propagation

c_0 - speed of propagation

$$\boxed{\bar{v} = c_0 \hat{a}}$$

$$f\left(t - \frac{\hat{a} \cdot \bar{r}}{c_0}\right)$$

$x=0$

the velocity vector is $\bar{v} = v_x \hat{a}_x + v_y \hat{a}_y + v_z \hat{a}_z$

with $c_0^2 = v_x^2 + v_y^2 + v_z^2$ $\quad \hat{a} = \dfrac{\bar{v}}{c_0}$

thus the model is

$$u(x,y,z,t) = f\left(t - \frac{\bar{v} \cdot \bar{r}}{c_0^2}\right)$$

$$= f\left(t - \frac{v_x x}{c_0^2} - \frac{v_y y}{c_0^2} - \frac{v_z z}{c_0^2}\right)$$

let $\bar{s} \triangleq \dfrac{\bar{v}}{c_0^2} = s_x \hat{a}_x + s_y \hat{a}_y + s_z \hat{a}_z$

$\therefore \quad s_x = \dfrac{v_x}{c_0^2} \quad s_y = \dfrac{v_y}{c_0^2} \quad s_z = \dfrac{v_z}{c_0^2}$

$$s_x^2 + s_y^2 + s_z^2 = \frac{1}{c_0^4}\left(v_x^2 + v_y^2 + v_z^2\right) = \frac{1}{c_0^2}$$

$\therefore$ we can write:

$$u(x,y,z,t) = f\left(t - s_x x - s_y y - s_z z\right)$$

$$\boxed{u(\bar{r},t) = f(t - \bar{s}\cdot\bar{r})}$$

Notice:

$$\partial_t^2 f(t-\bar{s}\cdot\bar{r}) = f''(t-\bar{s}\cdot\bar{r})$$

$$\partial_x^2 f(t-\bar{s}\cdot\bar{r}) = s_x^2 f''(t-\bar{s}\cdot\bar{r})$$

$$\partial_y^2 f(t-\bar{s}\cdot\bar{r}) = s_y^2 f''(t-\bar{s}\cdot\bar{r})$$

$$\partial_z^2 f(t-\bar{s}\cdot\bar{r}) = s_z^2 f''(t-\bar{s}\cdot\bar{r})$$

If we new S_x we could apply the one-way wave equation operator

$$\left(\partial_x - S_x \partial_t \right) u = 0$$

at the boundary $x=0$. This would model only outward propagating waves.

But consider our relation:

$$c_0^{-2} = S_x^2 + S_y^2 + S_z^2$$

$$\Longrightarrow \quad S_x = c_0^{-1} \left(1 - (c_0 S_y)^2 - (c_0 S_z)^2 \right)^{1/2}$$

using $\quad (1-x)^{1/2} \approx 1 - \frac{1}{2} x$

$$\left(1 - (c_0 S_y)^2 - (c_0 S_z)^2 \right)^{1/2} \approx 1 - \frac{1}{2} \left((c_0 S_y)^2 + (c_0 S_z)^2 \right)$$

$\therefore$ the one-way wave equation at $x=0$ can be written as:

$$\left\{ \partial_x - c_0^{-1} \left[1 - \frac{1}{2} \left((c_0 S_y)^2 + (c_0 S_z)^2 \right) \right] \partial_t \right\} u = 0$$

But we don't know S_y and S_z !

We could just stop at the 1^{st}-order approximation:

$$\boxed{\left(\partial_x - c_0^{-1} \partial_t \right) u = 0}$$

or, multiplying by c_0^{-1} :

$$\left[c_0^{-1}\partial_x - c_0^{-2}\left[1-\frac{1}{2}\left((c_0 s_y)^2+(c_0 s_z)^2\right)\right]\partial_t \right] u = 0$$

$$\left(c_0^{-1}\partial_x - \left[c_0^{-2} - \frac{1}{2}s_y^2 - \frac{1}{2}s_z^2 \right]\partial_t \right) u = 0$$

taking ∂_t :

$$\left(c_0^{-1}\partial_{xt} - \left[c_0^{-2} - \frac{1}{2}s_y^2 - \frac{1}{2}s_z \right]\partial_t^2 \right) u = 0$$

but using

$$\partial_y^2 f(t-\bar{s}\cdot\bar{r}) = s_y^2 f''(t-\bar{s}\cdot\bar{r}) = s_y^2\partial_t^2 f(t-\bar{s}\cdot\bar{r})$$

$$\partial_z^2 f(t-\bar{s}\cdot\bar{r}) = s_z^2\partial_t^2 f(t-\bar{s}\cdot\bar{r})$$

we get the second-order approximation:

$$\boxed{\left[c_0^{-1}\partial_{xt} - c_0^{-2}\partial_t^2 + \frac{1}{2}\partial_y^2 + \frac{1}{2}\partial_z^2 \right] u = 0}$$

Mur's Absorbing BCs (in 3D)

at the boundary $x=0$ of the mesh we have two approximate BCs:

$$\begin{cases} \underline{1^{st}\text{-Order}}: & (\partial_x - c_0^{-1}\partial_t)\, u(x,t) = 0 \\ \underline{2^{nd}\text{-Order}}: & \left(c_0^{-1}\partial_{xt} - c_0^{-2}\partial_t^2 + \frac{1}{2}\partial_y^2 + \frac{1}{2}\partial_z^2\right) u = 0 \end{cases}$$

Discretization of the $\underline{1^{st}\text{-Order}}$ approximation:

$$\partial_x u \approx \frac{1}{2}\left\{ \underbrace{\frac{u_1^n - u_0^n}{\Delta x}}_{\approx \partial_x u|_n} + \underbrace{\frac{u_1^{n+1} - u_0^{n+1}}{\Delta x}}_{\approx \partial_x u|_{n+1}} \right\}$$

$$\partial_t u \approx \frac{1}{2}\left\{ \underbrace{\frac{u_1^{n+1} - u_1^n}{\Delta t}}_{\approx \partial_t u|_1} + \underbrace{\frac{u_0^{n+1} - u_0^n}{\Delta t}}_{\approx \partial_t u|_0} \right\}$$

plugging into the first-order approximation

$$c_0\Delta t\left[\left(u_1^{n+1} - u_0^{n+1}\right) + \left(u_1^n - u_0^n\right)\right] - \Delta x\left[\left(u_1^{n+1} - u_1^n\right) + \left(u_0^{n+1} - u_0^n\right)\right] = 0$$

we're looking for an update equation for u_0^{n+1} so we should isolate that term on on side of the equation.

$$(C_0\Delta t + \Delta x)\, u_0^{n+1} = (C_0\Delta t + \Delta x)\, u_1^n$$

$$+ (C_0\Delta t - \Delta x)\left(u_1^{n+1} - u_0^n\right)$$

$$\therefore \boxed{u_0^{n+1} = u_1^n + \frac{(C_0\Delta t - \Delta x)}{(C_0\Delta t + \Delta x)}\left(u_1^{n+1} - u_0^n\right)}$$

Discretization of 2^{nd}-Order approximation:

we choose the point $x = \frac{1}{2}\Delta x$ $t = n\Delta t$ to apply second-order finite difference approximations.

$$\partial_y^2 u\Big|_{\frac{1}{2}}^{n} \approx \frac{1}{2}\left\{\frac{\left(u_{1j+1k}^n - 2u_{1jk}^n + u_{1j-1k}^n\right)}{\Delta y^2} + \frac{\left(u_{0j+1k}^n - 2u_{0jk}^n + u_{0j-1k}^n\right)}{\Delta y^2}\right\}$$

$$\partial_z^2 u\Big|_{\frac{1}{2}}^{n} \approx \frac{1}{2}\left\{\frac{\left(u_{1jk+1}^n - 2u_{1jk}^n + u_{1jk-1}^n\right)}{\Delta z^2} + \frac{\left(u_{0jk+1}^n - 2u_{0jk}^n + u_{0jk-1}^n\right)}{\Delta z^2}\right\}$$

$$\partial_t^2 u\Big|_{\frac{1}{2}}^{n} \approx \frac{1}{2}\left\{ \frac{\left(u_{1jk}^{n+1} - 2u_{1jk}^{n} + u_{1jk}^{n-1}\right)}{\Delta t^2} + \frac{\left(u_{0jk}^{n+1} - 2u_{0jk}^{n} + u_{0jk}^{n-1}\right)}{\Delta t^2} \right\}$$

$$\partial_{xt} u\Big|_{\frac{1}{2}}^{n} \approx \partial_t\left(\frac{u_{1jk}^{n} - u_{0jk}^{n}}{\Delta x}\right) \approx \frac{\frac{u_{1jk}^{n+1} - u_{1jk}^{n-1}}{2\Delta t} - \frac{u_{0jk}^{n+1} - u_{0jk}^{n-1}}{2\Delta t}}{\Delta x}$$

$$= \frac{u_{1jk}^{n+1} - u_{1jk}^{n-1} - u_{0jk}^{n+1} + u_{0jk}^{n-1}}{2\Delta t \Delta x}$$

We need to plug this into the 2nd-order approximation and isolate the u_{0jk}^{n+1} term.

write the second-order approximation as:

$$\left(2c_0\, \partial_{xt} - 2\partial_t^2 + c_0^2\, \partial_y^2 + c_0^2\, \partial_z^2\right) u = 0$$

because the term we want to isolate is only in the first two terms, we look at

$$\left(2c_0\, \partial_{xt} - 2\partial_t^2 + S\right) u = 0$$

where $S = c_0^2\, \partial_y^2 + c_0^2\, \partial_z^2$

$$\frac{\cancel{2} c_0 \left(u_{1jk}^{n+1} - u_{1jk}^{n-1} - \boxed{u_{0jk}^{n+1}} + u_{0jk}^{n-1} \right)}{\cancel{2}\, \Delta t \Delta x}$$

$$- \frac{1}{\Delta t^2} \left[\left(u_{1jk}^{n+1} - 2 u_{1jk}^{n} + u_{1jk}^{n-1} \right) + \left(\boxed{u_{0jk}^{n+1}} - 2 u_{0jk}^{n} + u_{0jk}^{n-1} \right) \right]$$

$$+ S = 0$$

See Mur's paper for the final expression.